Systems of Linear Equations

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Overview

What is a system of linear equations?

A collection of linear equations involving the same set variables.

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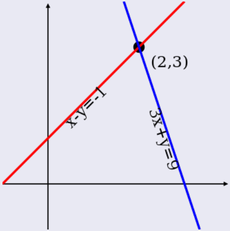
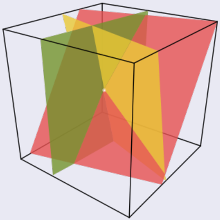
Example in 2 variables

$$\begin{cases} x - y + 1 = 0 \\ 3x + y - 9 = 0 \end{cases}$$



Example in 3 variables

$$\begin{cases} 3x + 2y - z - 1 = 0 \\ 2x - 2y + 4z + 2 = 0 \\ -x + \frac{1}{2}y - z = 0 \end{cases}$$



General form

System of m equations with n unknowns

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{cases}
$$

General form

System of m equations with n unknowns

Question: can this be written in matrix form?

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Answer: Yes. Store coefficients in A, unknowns in x, and translations in b.

General form

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The system can be written in matrix form as $Ax = b$

Types of systems

In $Ax = b$, A is a matrix of size m rows by n columns.

Choices of m and n affect solution sets

- $m = n$ and A is full rank (all m rows are linearly independent), the system has a unique solution.
- $m > n$, the system is **overdetermined**, i.e., there are more equations than variables. Overdetermined systems almost always have no solution (except when enough linear combinations of equations are present) .
- $m < n$, the system is **underdetermined**, i.e., there are fewer equations than variables. Underdetermined systems have either no solution or infinitely many.

Solving them in MATLAB

When in matrix form, $Ax = b$, the solution vector $x = A^{-1}b$. A^{-1} is the inverse of A, such that $A^{-1}A = AA^{-1} = I$, where I is the identity matrix.

Using matrix inverse

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x = inv(A) * b;
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Using matrix left division (Gaussian elimination)

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x = A\b;
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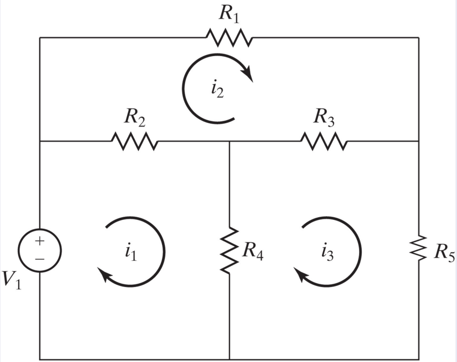
Real world problem

Electrical circuits

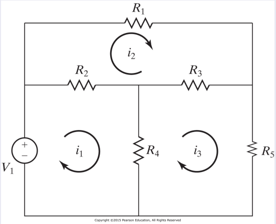


Find the currents i_1, i_2 and i_3, given the following:

- $\sum$ voltage around a circuit is 0.
- Voltage = Current x Resistance, $V = iR$.

Real world problem

Note: there are three circuits in this diagram.



Each circuit has a constraint: zero sum voltage

$$
\begin{cases}
-V_1 + R_2(i_1 - i_2) + R_4(i_1 - i_3) = 0 \\
R_1 i_2 + R_3(i_2 - i_3) + R_2(i_2 - i_1) = 0 \\
R_3(i_3 - i_2) + R_5 i_3 + R_4(i_3 - i_1) = 0
\end{cases}
$$

Rewriting the system

After slight algebraic manipulation:

$$\begin{cases} (R_2 + R_4)i_1 + (-R_2)i_2 + (-R_4)i_3 = V_1 \\ (-R_2)i_1 + (R_1 + R_2 + R_3)i_2 + (-R_3)i_3 = 0 \\ (-R_4)i_1 + (-R_3)i_2 + (R_3 + R_4 + R_5)i_3 = 0 \end{cases}$$

In-class exercise: given $R_1 = 2\Omega$, $R_2 = 4\Omega$, $R_3 = 6\Omega$, $R_4 = 8\Omega$, $R_5 = 10\Omega$ and $V_1 = 10V$, use MATLAB to solve for i_1, i_2 and i_3.