Introduction to Optimization

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Overview

What is optimization?

Optimization is a mathematical discipline concerned with finding the maxima and minima of functions, possibly subject to constraints.

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Optimization is a mathematical discipline concerned with finding the maxima and minima of functions, possibly subject to constraints.

Where is optimization used?

- Almost every Engineering discipline
- Architecture
- Nutrition
- Economics
- etc.

Overview

What do we optimize?

Most often, a real function of n variables:

$$f(x_1, x_2, ..., x_n) \in \mathcal{R}$$

Depending on discipline and context, this function is also known as:

- Cost function
- Objective function
- Loss function
- Utility function
- Reward function

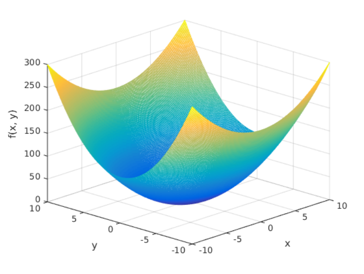
Two types of optimization:

- unconstrained
- constrained

Unconstrained

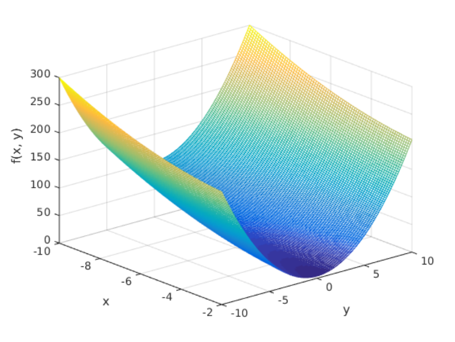
Example

$$\arg\min_{x,y} f(x,y) = x^2 + 2y^2 \quad (1)$$



Constrained example 1

$$\arg\min_{x,y} f(x,y) = x^2 + 2y^2$$
$$\text{subject to:} \tag{2}$$
$$x < 2$$



Constrained

Constrained example 2

$$\begin{aligned} \arg\min_{x,y} f(x,y) = x^2 + 2y^2 \\ \text{subject to:} \\ y < 2 \text{ and} \\ -2 < x < 5 \end{aligned} \tag{3}$$

MATLAB anonymous functions

Definition and Syntax

An anonymous function is a function that is not stored in a program file, but is associated with a variable whose data type is function_handle. Anonymous functions can accept inputs and return outputs, just as standard functions do. However, they can contain only a single executable statement. For example, to create an anonymous function that finds the square of a number:

```
>> sqr = @(x) x.^2;
>> sqr(2)

ans =

     4
```

MATLAB anonymous functions

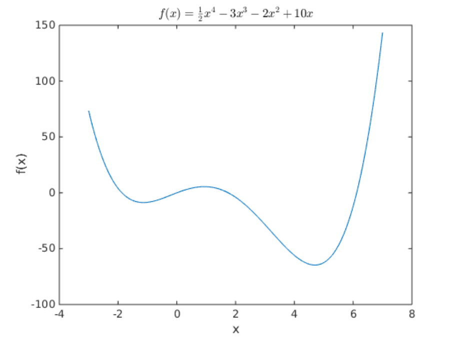
Function with two inputs

```
>> f = @(x, y) sin(x)*cos(y);
>> f(2, 4)

ans =

   -0.5944
```

MATLAB anonymous functions

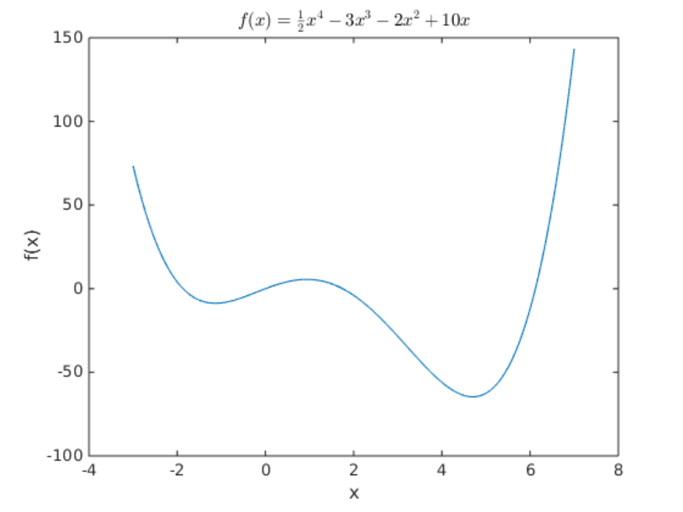
Plotting a simple polynomial

```
f = @(x) (0.5)*x.^4 - 3*x.^3 - 2*x.^2 + 10*x;
xs = linspace(-3, 7, 500);
ys = f(xs);
plot(xs, ys);
```



MATLAB anonymous functions

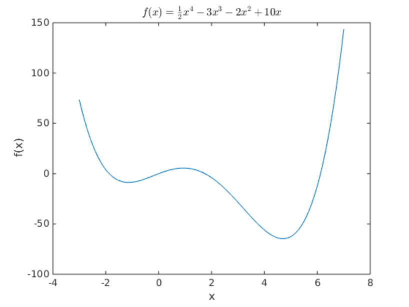
Where is the minima?

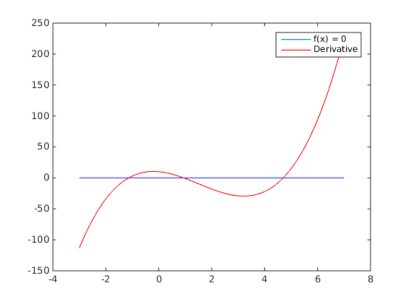


Derivative

$$f(x) = \frac{1}{2}x^4 - 3x^3 - 2x^2 + 10x \quad (4)$$

$$f'(x) = 2x^3 - 9x^2 - 4x + 10 \quad (5)$$





Numerical optimization

In this course we will look at **numerical** optimization (in contrast to analytical methods used in Calculus courses).

Numerical?

We do not know the mathematical formula for the function f we wish to optimize, but we can sample it.

Numerical optimization

When we don't know what f is, we can still sample

```
>> f(0.1)
ans =
   0.977050000000000
>> f(-1)
ans =
  -8.500000000000000
>> f(2)
ans =
    -4
```

A simple algorithm

- Decide on an interval [*low*, *high*]
- Sample x values of the function in that interval
- Pick the lowest value of the function on that interval as the minima

Numerical optimization

In MATLAB

```
domain = linspace(-3, 7, 500);
current_minima  = f(domain(1)); % default
min_x = domain(1); % default

for x = domain % loop over domain
    if( f(x) < current_minima )
        min_x = x; % update our estimate
        current_minima = f(x);
    end
end
% print out results
fprintf('The current_minima is at x = %.4f\n', min_x);
fprintf('At x=%.4f, f(x) = %.4f\n', min_x, current_minima);
```

Numerical optimization

Problems with the above approach? Assumptions, assumptions, assumptions...

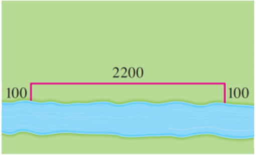
- Smoothness
- Is global minima in that domain?
- Is there more than one global minima?

Often, in practice, we settle for one solution, knowing there could be a better one.

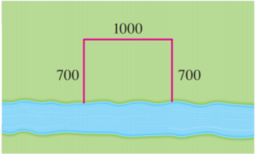
Constrained Optimization Example

A real world problem

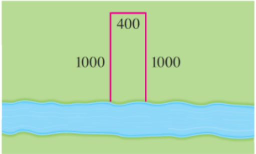
A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



Area = $100 \cdot 2200 = 220{,}000$ ft^2

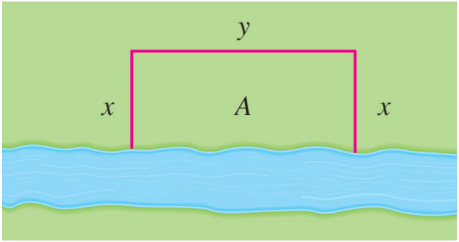


Area = $700 \cdot 1000 = 700{,}000$ ft^2



Area = $1000 \cdot 400 = 400{,}000$ ft^2

Fence



The general case

- Maximize $f(x, y) = A = xy$, subject to: $2x + y = 2400$

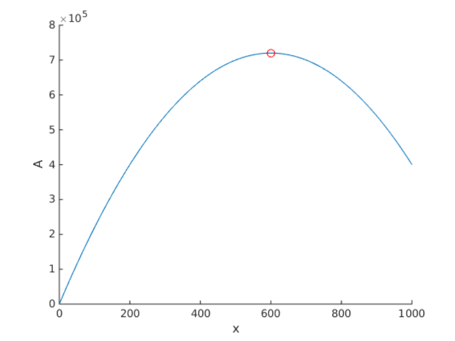
We first express A as a function of one variable by solving the constraint equation for y and substituting.

$2x + y = 2400 \implies y = 2400 - 2x$

$A = xy = x(2400 - 2x) = 2400x - 2x^2$

Fence

Plot of $A = 2400x - 2x^2$



Where is the area at a maximum?

$$\frac{dA}{dx} = 2400 - 4x$$

$$\frac{dA}{dx} = 0 \implies x = 600$$

Constraint

Visualizing the constraint $2x + y = 2400$