Basic Math Review for CS1340

Dr. Mihail

August 14, 2018

Sets

Definition of a set

A set is a collection of distinct objects, considered as an object in its own right. For example, the numbers 2, 4, and 6 are distinct objects when considered separately, but when they are considered collectively they form a single set of size three, written $\{2, 4, 6\}$. Sets are one of the most fundamental concepts in mathematics.

Have to know symbols

- $\in$: set membership. Example: $x \in \mathbb{R}$ is read x belongs to the set $\mathbb{R}$.
- $\cup$: union. Example: $X = A \cup B$ is read: X is the result of A union B, and contains **all** elements of A and B.
- $\cap$: intersection. Example $X = A \cap B$ is read X is the result of A intersect B, and contains elements that are in **BOTH** A and in B

Number sets

Naturals

- Natural numbers: $\mathbb{N}$

Number sets

Naturals

- Natural numbers: $\mathbb{N}$
- Examples: $0, 1, 2, 3, 4, \ldots$

Number sets

Naturals

- Natural numbers: $\mathbb{N}$
- Examples: $0, 1, 2, 3, 4, \dots$

Integers

- Integers: $\mathbb{Z}$

Number sets

Naturals

- Natural numbers: $\mathbb{N}$
- Examples: $0, 1, 2, 3, 4, ...$

Integers

- Integers: $\mathbb{Z}$
- Examples: $... -4, -3, -2, -1, 0, 1, 2, 3, 4, ...$

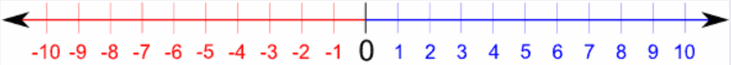
Number sets

Naturals

- Natural numbers: $\mathbb{N}$
- Examples: $0, 1, 2, 3, 4, ...$

Integers

- Integers: $\mathbb{Z}$
- Examples: $... -4, -3, -2, -1, 0, 1, 2, 3, 4, ...$



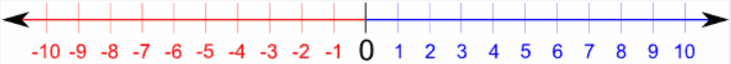
Number sets

Naturals

- Natural numbers: $\mathbb{N}$
- Examples: $0, 1, 2, 3, 4, \ldots$

Integers

- Integers: $\mathbb{Z}$
- Examples: $\ldots - 4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$



Rationals

- Rational numbers: $\mathbb{Q}$

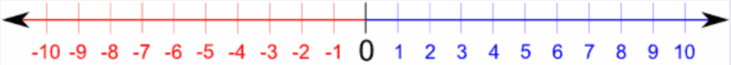
Number sets

Naturals

- Natural numbers: $\mathbb{N}$
- Examples: $0, 1, 2, 3, 4, \ldots$

Integers

- Integers: $\mathbb{Z}$
- Examples: $\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$



Rationals

- Rational numbers: $\mathbb{Q}$
- Examples: $\frac{1}{2}, \frac{2}{3}, -\frac{10}{7}, \frac{1}{3}$

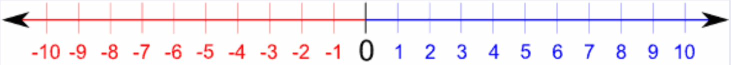
Number sets

Naturals

- Natural numbers: $\mathbb{N}$
- Examples: $0, 1, 2, 3, 4, \ldots$

Integers

- Integers: $\mathbb{Z}$
- Examples: $\ldots - 4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$

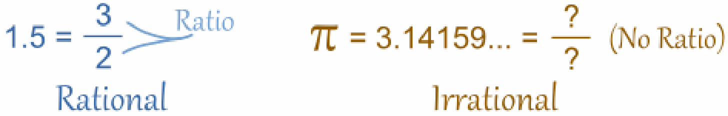


Rationals

- Rational numbers: $\mathbb{Q}$
- Examples: $\frac{1}{2}, \frac{2}{3}, -\frac{10}{7}, \frac{1}{3}$
- More generally, rational numbers are ratios of two whole numbers: $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ subject to b$\neq$ 0

Number sets contd.

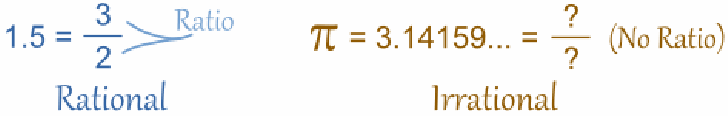
Irrationals



- Numbers that cannot be expressed as a ratio of two integers
- No set symbol, often noted as: $\mathbb{R} - \mathbb{Q}$
- Examples: $\pi, e, \sqrt{2}$

Number sets contd.

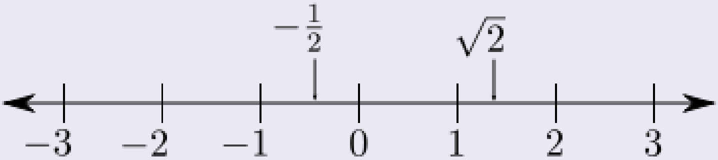
Irrationals



- Numbers that cannot be expressed as a ratio of two integers
- No set symbol, often noted as: $\mathbb{R} - \mathbb{Q}$
- Examples: $\pi, e, \sqrt{2}$

Reals

- Real numbers: $\mathbb{R}$



Number sets contd.

Imaginaries

- Imaginary numbers: $\mathbb{I}$
- They are numbers that, when squared, result in a negative number
- Example: $\sqrt{-9} = 3i$, because $(3i)^2 = -9$, here $i^2 = -1$

Number sets contd.

Imaginaries

- Imaginary numbers: $\mathbb{I}$
- They are numbers that, when squared, result in a negative number
- Example: $\sqrt{-9} = 3i$, because $(3i)^2 = -9$, here $i^2 = -1$

Algebraic numbers

- Algebraic numbers: $\mathbb{A}$
- Numbers that are roots (solutions) to at least one non-zero polynomial with rational coefficients
- Example: x in $2x^3 - 5x + 39$

Number sets contd.

Imaginaries

- Imaginary numbers: $\mathbb{I}$
- They are numbers that, when squared, result in a negative number
- Example: $\sqrt{-9} = 3i$, because $(3i)^2 = -9$, here $i^2 = -1$

Algebraic numbers

- Algebraic numbers: $\mathbb{A}$
- Numbers that are roots (solutions) to at least one non-zero polynomial with rational coefficients
- Example: x in $2x^3 - 5x + 39$

What about i

Is i also an algebraic number?

Number sets contd.

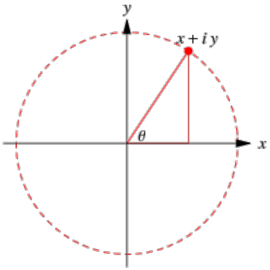
Complex

- Complex numbers: $\mathbb{C}$
- They are a combination of a real and an imaginary number
- Examples $10 - 2i, 2 + 3i$
- More generally, they have the form $x + iy$, where $x, y \in \mathbb{R}$

Number sets contd.

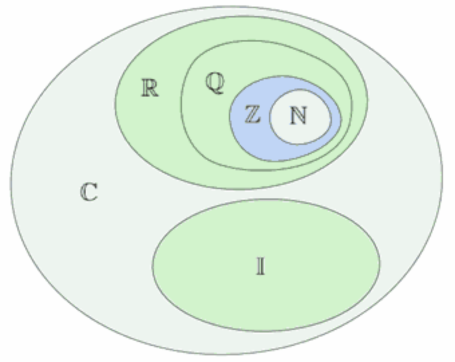
Complex

- Complex numbers: $\mathbb{C}$
- They are a combination of a real and an imaginary number
- Examples $10 - 2i, 2 + 3i$
- More generally, they have the form $x + iy$, where $x, y \in \mathbb{R}$



Operations on numbers

Venn diagram of number sets



Operations on numbers

Common operations

- Addition: $2 + 3 = 5$
- Subtraction $2 - 3 = -1$
- Multiplication $2 * 3 = 6$
- Division $\frac{2}{3} = 0.(6)$
- Exponentiation $2^3 = 8$

Variables

Variable may refer to:

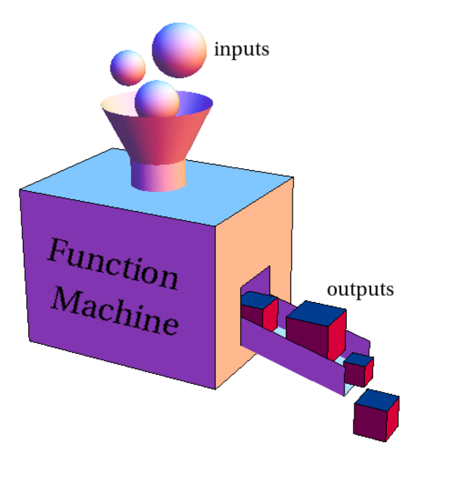
- In research: a logical set of attributes
- In mathematics: a **symbol** that represents a quantity in a mathematical expression
- In computer science: a **symbolic name** associated with a value and whose associated value may be changed

We shall use all 3 flavors in this course.

What is a function?

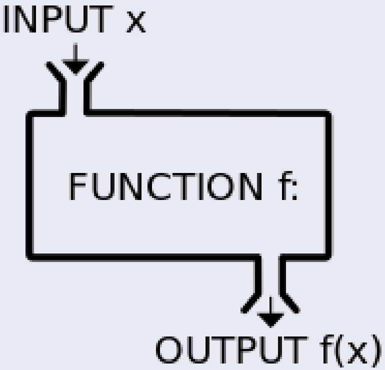
Functions

Intuition



Functions

Intuition useful for computer scientists

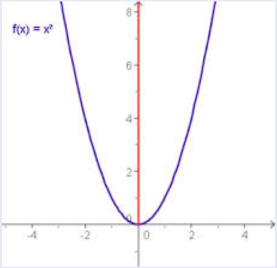


Functions

Informal definition

Think of a function as a "process" that takes input x and produces output f(x). For example, the function $f(x) = x^2$, takes an input x (a number) and "processes" it by squaring it.

Plotting a function with a single number as input



Terminology related to functions

Terms to **absolutely have to know**

- Function input: **domain**

Terminology related to functions

Terms to absolutely have to know

- Function input: **domain**
- Function output: **range** or more accurately **image**

Terminology related to functions

Terms to **absolutely have to know**

- Function input: **domain**
- Function output: **range** or more accurately **image**
- When plotting a function with scalar inputs, the X-axis is called the **abscissa**, the Y-axis is called the **ordinate**

Terminology related to functions

Terms to **absolutely have to know**

- Function input: **domain**
- Function output: **range** or more accurately **image**
- When plotting a function with scalar inputs, the X-axis is called the **abscissa**, the Y-axis is called the **ordinate**
- The input X, is also referred to as the **independent variable** or **predictor variable, regressor, controlled variable, manipulated variable, explanatory variable, etc.**

Terminology related to functions

Terms to **absolutely have to know**

- Function input: **domain**
- Function output: **range** or more accurately **image**
- When plotting a function with scalar inputs, the X-axis is called the **abscissa**, the Y-axis is called the **ordinate**
- The input X, is also referred to as the **independent variable** or **predictor variable, regressor, controlled variable, manipulated variable, explanatory variable, etc.**
- The output Y, is also referred to as the **dependent variable** or **response variable, regressand, measured variable, outcome variable, output variable, etc.**

Operations on functions

Composition

The idea is to "process" the input through one function, then use the result of that function as the input to the second. This results in a different function.

- Notation: given two functions f and g, the composition of g and f is written as $(g \circ f) = g(f(x))$.
- Example: if $f(x) = 2x + 3$, and $g(x) = x^2$, then $(g \circ f) = g(f(x)) = g(2x + 3) = (2x + 3)^2 = 4x^2 + 12x + 9$.
- $(f \circ g) \neq (g \circ f)$.

Operations on functions

Differentiation/Integration

Rates of change and areas under the curve.

- Derivative of a function f is often noted as f' or $\frac{d}{dx}[f(x)]$

Operations on functions

Differentiation/Integration

Rates of change and areas under the curve.

- Derivative of a function f is often noted as f' or $\frac{d}{dx}[f(x)]$
  - It is important to know if a function is **differentiable** and **where**

Operations on functions

Differentiation/Integration

Rates of change and areas under the curve.

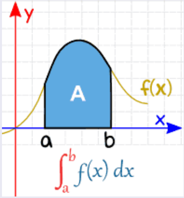
- Derivative of a function f is often noted as f' or $\frac{d}{dx}[f(x)]$
  - It is important to know if a function is **differentiable** and **where**
- Indefinite integral of a function f is written as $\int f(x)dx$
- Definite integral of a function f over an interval $[a, b]$ is written as $\int_a^b f(x)dx$

Operations on functions

Differentiation/Integration

Rates of change and areas under the curve.

- Derivative of a function f is often noted as f' or $\frac{d}{dx}[f(x)]$
  - It is important to know if a function is **differentiable** and **where**
- Indefinite integral of a function f is written as $\int f(x)dx$
- Definite integral of a function f over an interval $[a, b]$ is written as $\int_a^b f(x)dx$



Analytic/Numerical

In Calculus courses you were probably taught **analytic** solutions to differentiation and integration problems. In the real-world, you will most likely deal with numerical differentiation and integration. More on that later in the course.

Vector and Matrix Algebra

Scalars

A scalar is a simple quantity, or a number. For example, in $x = 1$, x is a scalar.

Vector and Matrix Algebra

Scalars

A scalar is a simple quantity, or a number. For example, in $x = 1$, x is a scalar.

Vectors

Going a bit further, a **vector** is **an ordered set of scalars**. For example, $[2, 3]$ is a vector.

Vector and Matrix Algebra

Scalars

A scalar is a simple quantity, or a number. For example, in $x = 1$, x is a scalar.

Vectors

Going a bit further, a **vector** is **an ordered set of scalars**. For example, $[2, 3]$ is a vector.

Vector elements

The position of the scalar in the ordered set is referred to as the **index**. In the example above, the index of the element 2 is 1, since it is the first element in the set. The index of 3 is 2, since it is the second element.

More about vectors

Vector dimensionality

- The number of elements a vector has is referred to as its **dimensionality**. For example, the vector $X = [x_1, x_2, x_3]$ has dimensionality 3, and if $x_1, x_2, x_3 \in \mathbb{R}$, then it is denoted as $X \in \mathbb{R}^3$.
- There can be any number dimensional vectors. For example 6-dimensional vectors $\in \mathbb{R}^6$.

More about vectors

Vector dimensionality

- The number of elements a vector has is referred to as its **dimensionality**. For example, the vector $X = [x_1, x_2, x_3]$ has dimensionality 3, and if $x_1, x_2, x_3 \in \mathbb{R}$, then it is denoted as $X \in \mathbb{R}^3$.
- There can be any number dimensional vectors. For example 6-dimensional vectors $\in \mathbb{R}^6$.

Vector magnitude

- A vector's magnitude is the distance (or L2-norm) from the origin of the space it "lives" in and a point. The magnitude is **computed** using the Pythagorean theorem (more accurately, a generalization of that known as Euclidian distance) using the following formula and notation:

$$|X| = \sqrt{(\sum_{i=1}^{n} x_i^2)} \qquad (1)$$

But I thought...

That...

- Vectors are*a mathematical object with a magnitude and direction*, not what you just told us.

But I thought...

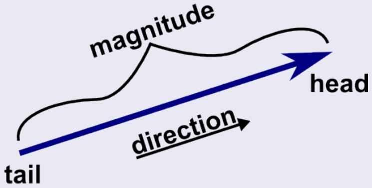
That...

- Vectors are*a mathematical object with a magnitude and direction*, not what you just told us.
- This definition is nothing but a special case of the definition in the previous slide.

But I thought...

That...

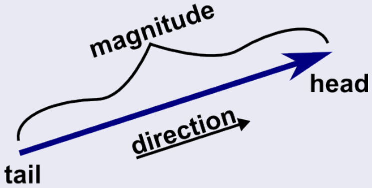
- Vectors are*a mathematical object with a magnitude and direction*, not what you just told us.
- This definition is nothing but a special case of the definition in the previous slide.



But I thought...

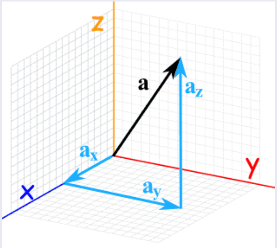
That...

- Vectors are*a mathematical object with a magnitude and direction*, not what you just told us.
- This definition is nothing but a special case of the definition in the previous slide.

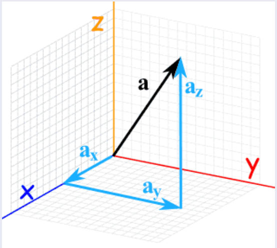


When the tail and the head are points on 2D plane, how can we compute magnitude?

3D visualization



3D visualization



In-class exercise

If $a = [1, 2, 3]$, what is $|a|$?

Matrices

Definition

A matrix is a rectangular table of numbers.

Matrices

Definition

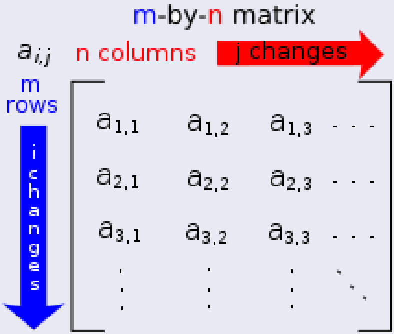
A matrix is a rectangular table of numbers.

Example

$$A = \begin{bmatrix} 4.86 & 0 & 0 & 0 & 0 & -2.60 & 0 & 0 & 0 & 0 \\ 0 & 5.13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.03 & 1.29 & 1.41 & 0 & 0 & 2.05 & 0 & 0.04 \\ 0 & 0 & 1.29 & 0.99 & 0 & 0 & 0 & 0 & 0 & -0.79 \\ 0 & 0 & 1.41 & 0 & 5.45 & 0 & 0 & 0 & 0 & 0 \\ -2.60 & 0 & 0 & 0 & 0 & 2.60 & 0.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.17 & 1.16 & 0 & 0 & 0 \\ 0 & 0 & 2.05 & 0 & 0 & 0 & 0 & 1.64 & 0 & 0.21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.48 & 0 \\ 0 & 0 & 0.04 & -0.79 & 0 & 0 & 0 & 0.21 & 0 & 4.21 \end{bmatrix}$$

Matrices

Structure in the 2D case



Matrices

Rows and Columns

- One can also think of a matrix as a collection of rows **or** a collection of columns.
- Or as a collection of **row vectors** or **column vectors**

Row/Column vectors

$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

- X has dimensionality 3x1, and is called a column vector
- Y has dimensionality 1x3, and is called a row vector

Matrices

Collection of column vectors

Given $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, we can form a matrix Z using X_1 and X_2:

$$Z = \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Collection of row vectors

Given $X_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$, we can form a matrix Z using X_1 and X_2:

$$Z = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Indexing

Say, $Z = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. The matrix is 2 rows by 3 colums (2x3).

Indexing

How can we address an element from a matrix?

Say, $Z = \begin{bmatrix} 1 & 2 & 3 \\ 4 & \color{red}{5} & 6 \end{bmatrix}$. The matrix is 2 rows by 3 colums (2x3).

Indexing

How can we address an element from a matrix?

Say, $Z = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. The matrix is 2 rows by 3 colums (2x3).

Simple

Each element has an assigned column and row number. Think of Z as follows:

$$Z = \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} \\ Z_{2,1} & Z_{2,2} & Z_{2,3} \end{bmatrix}$$

each $Z_{i,j}$ where $i \in \{\text{possible rows}\}$ and $j \in \{\text{possible columns}\}$, where possible rows for Z is the set $\{1, 2\}$ and the possible columns for Z is the set $\{1, 2, 3\}$.

Indexing

How can we address an element from a matrix?

Say, $Z = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. The matrix is 2 rows by 3 colums (2x3).

Simple

Each element has an assigned column and row number. Think of Z as follows:

$$Z = \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} \\ Z_{2,1} & Z_{2,2} & Z_{2,3} \end{bmatrix}$$

each $Z_{i,j}$ where $i \in \{\text{possible rows}\}$ and $j \in \{\text{possible columns}\}$, where possible rows for Z is the set $\{1, 2\}$ and the possible columns for Z is the set $\{1, 2, 3\}$.

"Where" is 5?

Second row, second column: $Z_{2,2}$

Operations on vectors and matrices

Addition and subtraction

If two matrices have the same dimensions r by c, including vectors and scalars as special cases, they can be added or subtracted by adding or subtracting the elements in the same positions in each matrix.

Operations on vectors and matrices

Addition and subtraction

If two matrices have the same dimensions r by c, including vectors and scalars as special cases, they can be added or subtracted by adding or subtracting the elements in the same positions in each matrix.

If A is r by c, and B is r by c, then for $C = A + B$, $C_{ij} = A_{ij} + B_{ij}$, similarly if $C = A - B$, $C_{ij} = A_{ij} - B_{ij}$.

Operations on vectors and matrices

Multiplication

Matrix multiplication summarizes a set of multiplications and additions.

- Multiplication of matrix by scalar: simply multiply each element of the matrix by the scalar.

Example: $a = 2$ and $X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then Ax or xA is a matrix formed as follows: $\begin{bmatrix} 2*1 & 2*2 & 2*3 \\ 2*4 & 2*5 & 2*6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$

Operations on vectors and matrices

Multiplication

Multiplication of two matrices:

- The two matrices must be **conformable**, that is if A is r_1 by c_1 and B is r_2 by c_2, then $C = A \times B$ is defined when $c_1 = r_2$ and C is of size r_1 by c_2.
- C_{ij} is found by multiplying each element of row i of A with each element of column j of B and adding up the multiplied pairs of real numbers.
- Exercises to follow as homework.

Operations on vectors and matrices

Transposition

The transpose of A, written as A^T is created by one the following ways:

- write the rows of A as the columns of A^T

Operations on vectors and matrices

Transposition

The transpose of A, written as A^T is created by one the following ways:

- write the rows of A as the columns of A^T
- write the columns of A as the rows of A^T

Transposition

The transpose of A, written as A^T is created by one the following ways:

- write the rows of A as the columns of A^T
- write the columns of A as the rows of A^T

Properties:

- $c^T = c$, if c is a scalar

Operations on vectors and matrices

Transposition

The transpose of A, written as A^T is created by one the following ways:

- write the rows of A as the columns of A^T
- write the columns of A as the rows of A^T

Properties:

- $c^T = c$, if c is a scalar
- $(A^T)^T = A$

Operations on vectors and matrices

Transposition

The transpose of A, written as A^T is created by one the following ways:

- write the rows of A as the columns of A^T
- write the columns of A as the rows of A^T

Properties:

- $c^T = c$, if c is a scalar
- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$

Operations on vectors and matrices

Transposition

The transpose of A, written as A^T is created by one the following ways:

- write the rows of A as the columns of A^T
- write the columns of A as the rows of A^T

Properties:

- $c^T = c$, if c is a scalar
- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Operations on vectors and matrices

Transposition

The transpose of A, written as A^T is created by one the following ways:

- write the rows of A as the columns of A^T
- write the columns of A as the rows of A^T

Properties:

- $c^T = c$, if c is a scalar
- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(cA)^T = cA^T$

On vectors

Transpose of a row vector results in a column vector. Transpose of a column vector results in a row vector.

Inner and outer products of vectors

Given two vectors with the same number of elements, e.g.: a and b both r by 1, we can define the inner and outer products as follows:

Inner product

$$a^T b = \sum_{i=1}^{r} a_i b_i \tag{2}$$

The inner product of a vector v with itself $v^T v$ is equal to the sums of squares of its elements, so has the property $v^T v \geq 0$.

Outer product

The outer product results in a matrix, of size r by r. If $O = ab^T$ is the outer product matrix, then $O_{ij} = a_i b_j$.

Special matrices

- Square matrices: have the same number of rows and columns

Special matrices

- Square matrices: have the same number of rows and columns
- Diagnoal matrices: square matrices that have all except the elements on the main diagonal equal to 0

Special matrices

- Square matrices: have the same number of rows and columns
- Diagnoal matrices: square matrices that have all except the elements on the main diagonal equal to 0
- Symmetric matrices: square matrices that have the same numbers above and below the main diagonal, i.e., a matrix A is symmetric if and only if $A_{ij} = Aji$.

Special matrices

- Square matrices: have the same number of rows and columns
- Diagnoal matrices: square matrices that have all except the elements on the main diagonal equal to 0
- Symmetric matrices: square matrices that have the same numbers above and below the main diagonal, i.e., a matrix A is symmetric if and only if $A_{ij} = Aji$.
- Identity matrix: diagonal matrix with all 1s on the main diagonal

Trace and determinants of square matrices

Trace

The trace of a square matrix is the sum of elements in its main diagonal. For a matrix A of size rxr, its trace, denoted as $Tr(A)$ is:

$$Tr(A) = \sum_{i=1}^{r} A_{ii}$$

Important property: $tr(A) = \sum_i \lambda_i$, where λ_i are the eigenvalues of matrix A.

Determinant

The determinant of a square matrix is a difficult calculation, but serves important purposes in optimization problems. Often, the sign is more important than its exact value. An important property is: $det(A) = \prod_i \lambda_i$