Basic Math Review for CS4830

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Sets

Definition of a set

A set is a collection of distinct objects, considered as an object in its own right. For example, the numbers 2, 4, and 6 are distinct objects when considered separately, but when they are considered collectively they form a single set of size three, written $\{2, 4, 6\}$. Sets are one of the most fundamental concepts in mathematics.

Have to know symbols

- $\in$: set membership. Example: $x \in \mathbb{R}$ is read x belongs to the set $\mathbb{R}$.
- $\cup$: union. Example: $X = A \cup B$ is read: X is the result of A union B, and contains **all** elements of A and B.
- $\cap$: intersection. Example $X = A \cap B$ is read X is the result of A intersect B, and contains elements that are in **BOTH** A and in B

Number sets

Naturals

- Natural numbers: $\mathbb{N}$

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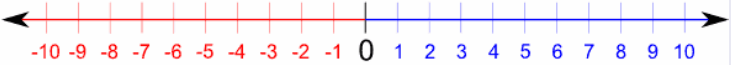

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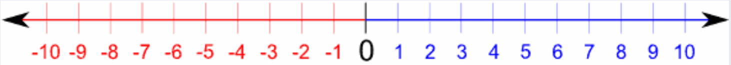

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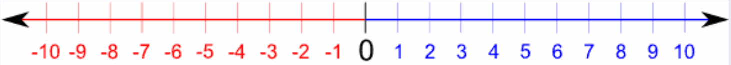

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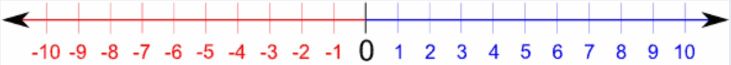
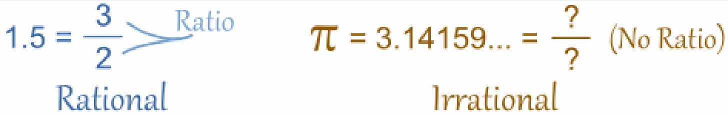


Rationals

- Rational numbers: $\mathbb{Q}$
- Examples: $\frac{1}{2}, \frac{2}{3}, -\frac{10}{7}, \frac{1}{3}$
- More generally, rational numbers are ratios of two whole numbers: $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ subject to b$\neq$ 0

Number sets contd.

Irrationals



- Numbers that cannot be expressed as a ratio of two integers
- No set symbol, often noted as: $\mathbb{R} - \mathbb{Q}$
- Examples: π, e, $\sqrt{2}$

Number sets contd.

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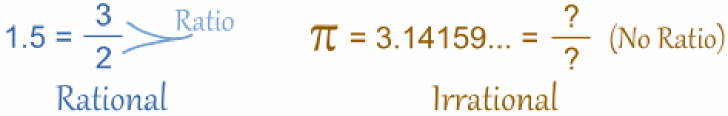

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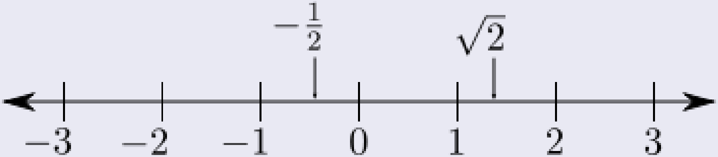
Reals

- Real numbers: $\mathbb{R}$



Number sets contd.

Imaginaries

- Imaginary numbers: $\mathbb{I}$
- They are numbers that, when squared, result in a negative number
- Example: $\sqrt{-9} = 3i$, because $(3i)^2 = -9$, here $i^2 = -1$

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Algebraic numbers

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What about i

Is i also an algebraic number?

Number sets contd.

Complex

- Complex numbers: $\mathbb{C}$
- They are a combination of a real and an imaginary number
- Examples $10 - 2i, 2 + 3i$
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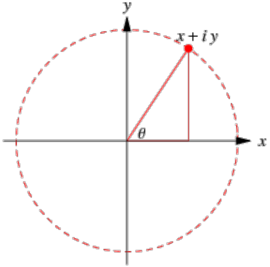
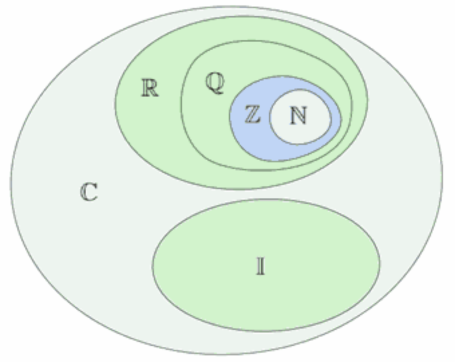


Operations on numbers

Venn diagram of number sets



Operations on numbers

Common operations

- Addition: $2 + 3 = 5$
- Subtraction $2 - 3 = -1$
- Multiplication $2 * 3 = 6$
- Division $\frac{2}{3} = 0.(6)$
- Exponentiation $2^3 = 8$

Variables

Variable may refer to:

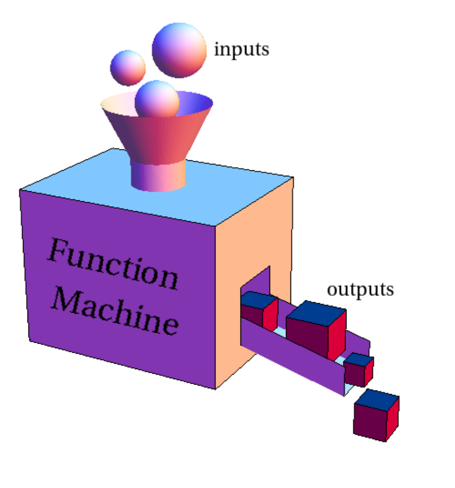
- In research: a logical set of attributes
- In mathematics: a **symbol** that represents a quantity in a mathematical expression
- In computer science: a **symbolic name** associated with a value and whose associated value may be changed

We shall use all 3 flavors in this course.

What is a function?

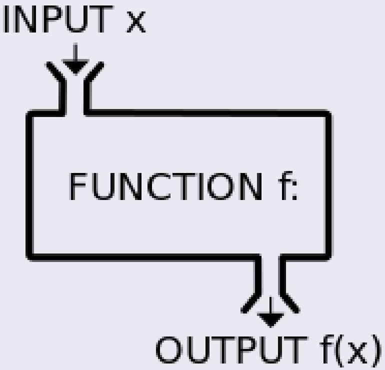
Functions

Intuition



Functions

Intuition useful for computer scientists

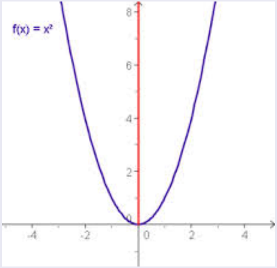


Functions

Informal definition

Think of a function as a "process" that takes input x and produces output f(x). For example, the function $f(x) = x^2$, takes an input x (a number) and "processes" it by squaring it.

Plotting a function with a single number as input



Terminology related to functions

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Operations on functions

Composition

The idea is to "process" the input through one function, then use the result of that function as the input to the second. This results in a different function.

- Notation: given two functions f and g, the composition of g and f is written as $(g \circ f) = g(f(x))$.
- Example: if $f(x) = 2x + 3$, and $g(x) = x^2$, then $(g \circ f) = g(f(x)) = g(2x + 3) = (2x + 3)^2 = 4x^2 + 12x + 9$.
- $(f \circ g) \neq (g \circ f)$.

Operations on functions

Differentiation/Integration

Rates of change and areas under the curve.

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- Indefinite integral of a function f is written as $\int f(x)dx$
- Definite integral of a function f over an interval $[a, b]$ is written as $\int_a^b f(x)dx$

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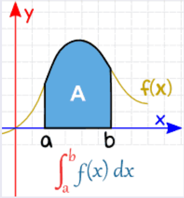

Analytic/Numerical

In Calculus courses you were probably taught **analytic** solutions to differentiation and integration problems. In the real-world, you will most likely deal with numerical differentiation and integration. More on that later in the course.

Vector and Matrix Algebra

Scalars

A scalar is a simple quantity, or a number. For example, in $x = 1$, x is a scalar.

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Going a bit further, a **vector** is **an ordered set of scalars**. For example, $[2, 3]$ is a vector.

Vector elements

The position of the scalar in the ordered set is referred to as the **index**. In the example above, the index of the element 2 is 1, since it is the first element in the set. The index of 3 is 2, since it is the second element.

More about vectors

Vector dimensionality

- The number of elements a vector has is referred to as its **dimensionality**. For example, the vector $X = [x_1, x_2, x_3]$ has dimensionality 3, and if $x_1, x_2, x_3 \in \mathbb{R}$, then it is denoted as $X \in \mathbb{R}^3$.
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Vector magnitude

- A vector's magnitude is the distance (or L2-norm) from the origin of the space it "lives" in and a point. The magnitude is **computed** using the Pythagorean theorem (more accurately, a generalization of that known as Euclidian distance) using the following formula and notation:

$$|X| = \sqrt{(\sum_{i=1}^{n} x_i^2)} \tag{1}$$

But I thought...

That...

- Vectors are*a mathematical object with a magnitude and direction*, not what you just told us.

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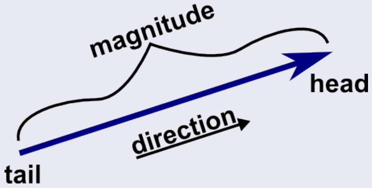

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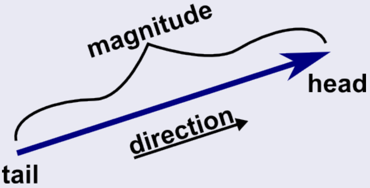
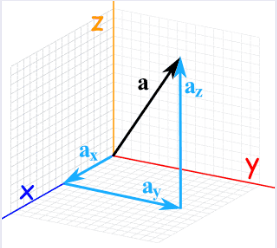


When the tail and the head are points on 2D plane, how can we compute magnitude?

3D visualization



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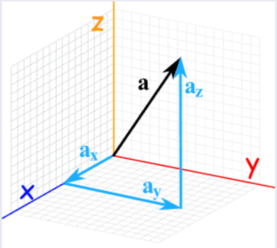

In-class exercise

If $a = [1, 2, 3]$, what is $|a|$?

Matrices

Definition

A matrix is a rectangular table of numbers.

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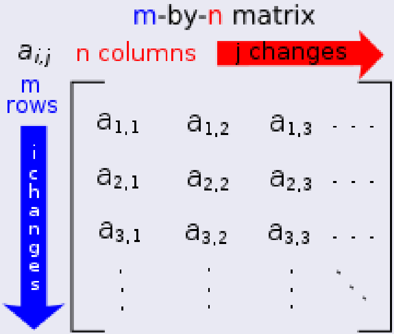
Example

$$A = \begin{bmatrix} 4.86 & 0 & 0 & 0 & 0 & -2.60 & 0 & 0 & 0 & 0 \\ 0 & 5.13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.03 & 1.29 & 1.41 & 0 & 0 & 2.05 & 0 & 0.04 \\ 0 & 0 & 1.29 & 0.99 & 0 & 0 & 0 & 0 & 0 & -0.79 \\ 0 & 0 & 1.41 & 0 & 5.45 & 0 & 0 & 0 & 0 & 0 \\ -2.60 & 0 & 0 & 0 & 0 & 2.60 & 0.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.17 & 1.16 & 0 & 0 & 0 \\ 0 & 0 & 2.05 & 0 & 0 & 0 & 0 & 1.64 & 0 & 0.21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.48 & 0 \\ 0 & 0 & 0.04 & -0.79 & 0 & 0 & 0 & 0.21 & 0 & 4.21 \end{bmatrix}$$

Matrices

Structure in the 2D case



Matrices

Rows and Columns

- One can also think of a matrix as a collection of rows **or** a collection of columns.
- Or as a collection of **row vectors** or **column vectors**

Row/Column vectors

$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

- X has dimensionality 3x1, and is called a column vector
- Y has dimensionality 1x3, and is called a row vector

Matrices

Collection of column vectors

Given $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, we can form a matrix Z using X_1 and X_2:

$$Z = \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Collection of row vectors

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$$Z = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Indexing

Say, $Z = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. The matrix is 2 rows by 3 colums (2x3).

Indexing

How can we address an element from a matrix?

Say, $Z = \begin{bmatrix} 1 & 2 & 3 \\ 4 & \color{red}{5} & 6 \end{bmatrix}$. The matrix is 2 rows by 3 colums (2x3).

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Simple

Each element has an assigned column and row number. Think of Z as follows:

$$Z = \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} \\ Z_{2,1} & Z_{2,2} & Z_{2,3} \end{bmatrix}$$

each $Z_{i,j}$ where $i \in \{\text{possible rows}\}$ and $j \in \{\text{possible columns}\}$, where possible rows for Z is the set $\{1, 2\}$ and the possible columns for Z is the set $\{1, 2, 3\}$.

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"Where" is 5?

Second row, second column: $Z_{2,2}$

Operations on vectors and matrices

Addition and subtraction

If two matrices have the same dimensions r by c, including vectors and scalars as special cases, they can be added or subtracted by adding or subtracting the elements in the same positions in each matrix.

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If A is r by c, and B is r by c, then for $C = A + B$, $C_{ij} = A_{ij} + B_{ij}$, similarly if $C = A - B$, $C_{ij} = A_{ij} - B_{ij}$.

Operations on vectors and matrices

Multiplication

Matrix multiplication summarizes a set of multiplications and additions.

- Multiplication of matrix by scalar: simply multiply each element of the matrix by the scalar.

Example: $a = 2$ and $X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then Ax or xA is a matrix formed as follows: $\begin{bmatrix} 2*1 & 2*2 & 2*3 \\ 2*4 & 2*5 & 2*6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$

Operations on vectors and matrices

Multiplication

Multiplication of two matrices:

- The two matrices must be **conformable**, that is if A is r_1 by c_1 and B is r_2 by c_2, then $C = A \times B$ is defined when $c_1 = r_2$ and C is of size r_1 by c_2.
- C_{ij} is found by multiplying each element of row i of A with each element of column j of B and adding up the multiplied pairs of real numbers.

Transposition

The transpose of A, written as A^T is created by one the following ways:

- write the rows of A as the columns of A^T

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Properties:

- $c^T = c$, if c is a scalar

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- $(cA)^T = cA^T$

On vectors

Transpose of a row vector results in a column vector. Transpose of a column vector results in a row vector.

Inner and outer products of vectors

Given two vectors with the same number of elements, e.g.: a and b both r by 1, we can define the inner and outer products as follows:

Inner product

$$a^T b = \sum_{i=1}^{r} a_i b_i \tag{2}$$

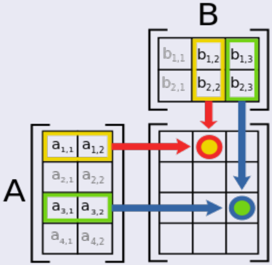
The inner product of a vector v with itself $v^T v$ is equal to the sums of squares of its elements, so has the property $v^T v \geq 0$.

Outer product

The outer product results in a matrix, of size r by r. If $O = ab^T$ is the outer product matrix, then $O_{ij} = a_i b_j$.

Operations on vectors and matrices

Multiplication



- Elements in the resulting matrix $M = A * B$, $M_{ij} = dot(A[i, *], B[*, j])$, where $*$ indicates all possible rows/colummns, and *dot* is the inner product (results in a scalar).
- $A[i, *]$ is a row vector, $B[*, j]$ is a column vector.

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- Identity matrix: diagonal matrix with all 1s on the main diagonal

Trace and determinants of square matrices

Trace

The trace of a square matrix is the sum of elements in its main diagonal. For a matrix A of size rxr, its trace, denoted as $Tr(A)$ is:

$$Tr(A) = \sum_{i=1}^{r} A_{ii}$$

Important property: $tr(A) = \sum_i \lambda_i$, where λ_i are the eigenvalues of matrix A.

Determinant

The determinant of a square matrix is a difficult calculation, but serves important purposes in optimization problems. Often, the sign is more important than its exact value. An important property is: $det(A) = \prod_i \lambda_i$